

Corrigendum to “Generators of the Hecke algebra of (S_{2n}, B_n) ”

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Abstract

In [1], among other things, we observed that the structure constants of the Hecke algebra of the Gel'fand pair (S_{2n}, B_n) are polynomials in n . It is brought to attention by Omar Tout that there is a missing argument in its proof. Here we provide the details of the missing argument by further analyzing various actions of the hyperoctahedral group.

Keywords: Farahat-Higman rings, structure constants, B_n -conjugacy classes

1 Introduction

The hyperoctahedral group B_n is the centralizer of the permutation

$$t_n = (12)(34) \cdots (2n-1\ 2n) \quad (1)$$

in the symmetric group S_{2n} . Here, $(2i-1\ 2i)$ stands for the cycle that interchanges $2i-1$ with $2i$. The Hecke algebra of the pair (S_{2n}, B_n) , denoted by H_n , is the convolution algebra of integer valued functions on S_{2n} that are constant on the double-cosets $B_n x B_n$, $x \in S_{2n}$.

Let x_1, \dots, x_r be a full list of representatives for the B_n -double cosets in S_{2n} , and let $\chi_i(n)$ ($i = 1, \dots, r$) denote the corresponding characteristic function on $\bar{x}_i := B_n x_i B_n$. Clearly, $\chi_1(n), \dots, \chi_r(n)$ form a \mathbb{Z} -basis for H_n , and therefore, for each i and j from $\{1, \dots, r\}$ there exist unique integers $b_{ij}^1(n), \dots, b_{ij}^r(n) \in \mathbb{Z}$ such that

$$\chi_i(n) * \chi_j(n) = \sum_{k=1}^r b_{ij}^k(n) \chi_k(n). \quad (2)$$

Our purpose in this paper is to provide a missing argument from the proof of the fact [Theorem 4.2, [1]] that, for all sufficiently large $n \in \mathbb{Z}$, and $k = 1, \dots, r$, the structure constants defined by the eqn. (2) are of the form

$$b_{ij}^k(n) = 2^{n-\alpha_k} n! f_{ij}^k(n),$$

where $\alpha_k \in \mathbb{Z}$ is a constant, and $f_{ij}^k(n) \in \mathbb{Z}[n]$ is a polynomial. The exact argument that is used in [1] and the demonstration of its failure is explained in more detail in the sequel, however, we give its synopsis here.

We view H_n as a subalgebra of the group ring $\mathbb{Z}[S_{2n}]$ by identifying $f \in H_n$ with the sum

$$f \rightsquigarrow \sum_{x \in S_{2n}} f(x)x \in \mathbb{Z}[S_{2n}].$$

Accordingly, the convolution product of two functions $f, g \in H_n$ translates to the ordinary product

$$f * g \rightsquigarrow \sum_{x, y \in S_{2n}} f(x)g(y)xy \in \mathbb{Z}[S_{2n}].$$

As the coefficients of the elements of \bar{x}_k in the expansion of $\chi_i(n) * \chi_j(n)$ are the same, it follows that $b_{ij}^k(n)$ is equal to number of couples $(a, b) \in \bar{x}_i \times \bar{x}_j$ such that $ab = x_k$. For subsets A, B and C of S_{2n} we denote $\{(a, b) \in A \times B : ab \in C\}$ by $V(A \times B; C)$, and we observe:

$$b_{ij}^k(n)|\bar{x}_k| = |V(\bar{x}_i \times \bar{x}_j; \bar{x}_k)|.$$

In [1], the cardinalities $|\bar{x}_k|$ and $|V(\bar{x}_i \times \bar{x}_j; \bar{x}_k)|$ are calculated under the assumption that the sets $V(\bar{x}_i \times \bar{x}_j; \bar{x}_k)$ grow uniformly as n gets bigger. It is brought to our attention by Omar Tout that this assumption is not true. In this paper, we amend this problem by replacing \bar{x}_k with a suitable subset of it. At the same time, this fixture discloses a subtle relationship between B_∞ -conjugacy classes in S_{2n} and B_n -double cosets.

2 Preliminaries

In this section we introduce some new notation in addition to what have from [1].

2.1 Partitions and permutations

Although we preserve the background from [1], we briefly recall the basic notation for partitions. A partition is a finite, non-increasing sequence of integers. The set of all partitions is denoted by \mathcal{P} . If a partition is obtained from another by adding or removing a finite number of zeros to or from the tail, we call these two partitions equivalent. Clearly, this defines an equivalence relation and we identify the elements of \mathcal{P} with each other according to this relation. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition. Then

1. Any non-zero entry λ_i is called a *part* of λ . The multi-set of parts of λ is denoted by $p(\lambda)$.
2. The integer $|p(\lambda)|$ is called the *length* of λ and denoted by $l(\lambda)$.
3. The integer $\sum_{\lambda_i \in p(\lambda)} \lambda_i$ is called the *size* of λ and denoted by $|\lambda|$.
4. The *weight* $w(\lambda)$ is defined to be the integer $l(\lambda) + |\lambda|$.

Let $\lambda, \mu \in \mathcal{P}$ be two partitions. If μ is a subsequence of λ , then we write $\mu \subseteq \lambda$. In this case, the partition obtained from λ by removing the elements of μ is denoted by $\lambda - \mu$. $\lambda + \mu$ is the partition obtained by vector addition of the original partitions. The unique partition that is obtained by reordering into a sequence of the union of multi-sets $p(\lambda)$ and $p(\mu)$ is denoted by $\lambda \cup \mu$. The exponential notation for a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ is $\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \dots)$, where $m_i(\lambda)$ is the number occurrence of i in $p(\lambda)$. Given a positive integer $n > w(\lambda)$, the *n-completion* $\lambda(n)$ of λ is the partition $\lambda \cup (1^{n-|\lambda|})$. Observe that $l(\lambda(n)) = l(\lambda) + n - |\lambda|$.

For permutations, when it is needed we write $(i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_r)$ in place of a cycle $(i_1 \dots i_r)$.

2.2 Infinite symmetric groups

As usual, the notation $S_\infty, S_{2\infty}$, and B_∞ stand for the direct limits of the systems $S_n \hookrightarrow S_{n+1}$, $S_{2n} \hookrightarrow S_{2n+2}$, and $B_n \hookrightarrow B_{n+1}$, $n = 1, 2, \dots$, which are directed by the obvious embeddings. In particular, an element $x \in S_\infty$ is a set automorphism on \mathbb{N} , the set of non-negative integers, such that $x(i) \neq i$ for only finitely many $i \in \mathbb{N}$.

We use the notation \mathbb{X} to denote the set of all 2-element subsets of \mathbb{N} . We call an element $D_i \in \mathbb{X}$ a *couple*, if it is of the form $D_i := \{2i - 1, 2i\} \in \mathbb{X}$ for some $i \in \mathbb{N}$. The integers contained in the same couple are said to be *partners* to each other, and the partner of a number $k \in D_i$ is denoted by $t(k)$. Obviously, $t(k) = t_n(k)$ for all $n \geq k/2$, where t_n is as in (1). For a subset S of \mathbb{N} , the set $t(S)$ is defined in the expected way. Define

1. $\mathbb{X}(n) := \{\{i, j\} \in \mathbb{X} : i, j \leq 2n\}$,
2. $\mathbb{D}(n) := \{D_i : i = 1, 2, \dots, n\}$,
3. $\mathbb{D} := \{D_i : i = 1, 2, \dots\}$.

The infinite symmetric group S_∞ (as well as $S_{2\infty}$) acts on \mathbb{X} by

$$x \cdot \{i, j\} = \{x(i), x(j)\}, \quad x \in S_\infty, \{i, j\} \in \mathbb{X}.$$

Remark 2.1. Observe that $x \in S_{2\infty}$ lies in B_∞ if and only if it stabilizes the subset $\mathbb{D} \subseteq \mathbb{X}$. Similarly, $x \in S_{2n}$ lies in B_n if and only if it stabilizes the subset $\mathbb{D}(n) \subseteq \mathbb{X}(n)$.

2.3 Support

The *support* of two elements $x, y \in S_\infty$ is defined to be $S(x, y) = S(x) \cup S(y)$, where $S(x) = \{i \in \mathbb{N} : x(i) \neq i\}$. Recall

Lemma 2.2. For $x, y \in S_\infty$, there exists $a \in S_\infty$ such that $(axa^{-1}, aya^{-1}) \in S_n \times S_n$ if and only if $|S(x, y)| \leq n$.

Proof. See [2]. □

Next, we introduce some useful variants of the notion of support.

Definition 2.3. The \mathbb{D} -support $D(x)$ of an element $x \in S_{2n} \subseteq S_{2\infty}$ is

$$D(x) := \{D_i \in \mathbb{D} : x(D_i) \notin \mathbb{D}\},$$

and the *unpaired* \mathbb{D} -support $DS(x)$ of x is

$$DS(x) := \bigcup_{D_i \in D(x)} D_i.$$

Paraphrasing Definition 2.3; the \mathbb{D} -support of x is the set of all couples that are mapped to non-couples, and the unpaired \mathbb{D} -support of x is the set of all partners that are mapped to non-partners.

Example 2.4. Let $x = (12)$, $y = (132)$ and $z = (13245)$ be three permutations that are written in cycle notation. Then we have

$$\begin{aligned}\emptyset = DS(x) &\subsetneq S(x) = \{1, 2\} \\ \{1, 2, 3, 4\} = DS(y) &\supsetneq S(y) = \{1, 2, 3\}, \\ \{3, 4, 5, 6\} = DS(z) &\neq S(z) = \{1, 2, 3, 4, 5\},\end{aligned}$$

which shows that there is no uniform containment relation between the unpaired \mathbb{D} -support and the (ordinary) support.

Some of relations between different types of support is revealed by our next result.

Lemma 2.5. Let $x \in S_{2n}$. Then

1. $2|D(x)| = |DS(x)|$;
2. For any $y \in B_n x B_n$ we have $2|D(x)| = |DS(x)| = |DS(y)| = 2|D(y)|$;
3. There exist $y \in B_n x B_n$ such that $D(x) = D(y)$ and $S(y) = DS(y)$;

Proof. The first assertion is obvious, and the second follows from Remark 2.1. To prove the third part of the lemma, we start with a claim:

If for some $j \in \mathbb{N}$, $D_j \cap S(x) \neq \emptyset$ and $D_j \notin D(x)$, then there exists $b = b_j \in B_n$ such that $D_j \cap S(bx) = \emptyset$ and $D(x) = D(bx)$.

Proof of the claim. First assume that $x(2j-1) \neq 2j$. Since $D_j \notin D(x)$, the set $\{x(2j-1), x(2j)\}$ is equal to some $D_i \neq D_j$. We then set

$$b = (2j-1 \rightarrow x(2j-1))(2j \rightarrow x(2j)). \quad (3)$$

It is straightforward to verify that

1. $b \in B_n$, and
2. $bx(D_j) = D_j$, hence $D_j \notin D(bx)$, and $D_j \cap S(bx) = \emptyset$.

Next we assume that $x(2j-1) = 2j$. Since $D_j \notin D(x)$, we know that there exists $D_i \in \mathbb{D}$ such that $x(D_j) = D_i$. Since the partner of $2j-1$ is $2j$, we see that $i = j$ and that $x(2j) = 2j-1$. In this case, we set $b = (2j-1 \rightarrow 2j) \in B_n$. It is straightforward to verify that $D_j \notin D(bx)$, and $D_j \cap S(bx) = \emptyset$. This finishes the proof of our claim. \square

Now, by applying $b = b_j$ to x for each j as in our claim, we arrive at an element $y \in B_n x$ such that $D(x) = D(y)$, and if $D_j \notin D(y)$ for some $j \in \mathbb{N}$, then $D_j \cap S(y) = \emptyset$. Equivalently, there exists $y \in B_n x$ such that $D(x) = D(y)$ and $S(y) \subseteq DS(y) = \bigcup_{D_i \in D(y)} D_i$.

It remains to show that if $j \in DS(y) \setminus S(y)$, then there exist $b \in B_n$ such that $j \in S(yb) \cap DS(yb)$ and $D(yb) = D(y)$. Indeed, for $b = (j \rightarrow t(j))$, we have $D(yb) = D(y)$. Moreover, since $yb(j) = y(t(j)) \neq j$ (as $y(j) = j$), j is a member of $S(yb)$. On the other hand, it is straightforward to check that the couple $\{j, t(j)\}$ is an element of $D(yb)$. In particular we see that $j \in DS(yb)$. Hence $j \in DS(yb) \cap S(yb)$. The proof is complete. \square

Definition 2.6. For a pair of elements $x, y \in S_{2\infty}$, the \mathbb{D} -support and the *unpaired* \mathbb{D} -support are defined, respectively, by

$$\begin{aligned} D(x, y) &= D(x) \cup D(y), \\ DS(x, y) &= DS(x) \cup DS(y). \end{aligned}$$

The *completed support* $CS(x, y)$ of (x, y) is defined to be

$$CS(x, y) = S(xy) \cup t(S(xy)) \cup DS(x, y).$$

Lemma 2.7. Let $(x, y) \in S_{2\infty} \times S_{2\infty}$. Then $t(CS(x, y)) = CS(x, y)$ and for an integer $i \notin CS(x, y)$ the following hold:

1. If $y(i) = j$ if and only if $x(j) = i$;
2. $i \in S(x)$ if and only if $i \in S(y)$;
3. If $y(i) = j$ then $y(t(i)) = t(j)$ and $x(t(j)) = t(x(j))$.

Proof. By definition of the completed support it is clear that $t(CS(x, y)) = CS(x, y)$. Observe that $xy(i) = i$ for $i \notin CS(x, y)$.

1. Let j be such that $y(i) = j$. Then $i = xy(i) = x(j)$, which proves the first assertion.
2. Suppose $i \in S(y)$. Then by Part 1., if $y(i) = j$, then $x(j) = i$ which shows that $i \in S(x)$. Conversely, suppose $i \in S(x)$. Then there exist $j \neq i$ such that $x(j) = i = xy(i)$, which means $y(i) = j$.
3. It follows from the fact that CS is closed under the action of t , $\{i, t(i)\} \cap CS(x, y) = \emptyset$. In particular $i, t(i) \notin DS(y)$. This means that $\{y(i), y(t(i))\}$ is couple, which proves the claim. As for the second part, we know that $t(i) \notin CS(x, y)$, and that $y(t(i)) = t(j)$. Thus, by Part 1, $x(t(j)) = t(i)$.

□

2.4 Indexing the conjugacy classes

Suppose that the cycle decomposition (with singletons included) of a permutation $x \in S_n$ is given by

$$x = c_1 \cdots c_k. \tag{4}$$

Customarily, the partition $\lambda_{(x)}$ defined by the positive integers $|S(c_1)|, \dots, |S(c_k)|$ is called the cycle type of x . The *stable cycle type* λ_x of x is defined by setting:

$$\lambda_x = \lambda_{(x)} - (1^{l(\lambda_{(x)})}). \tag{5}$$

The stable cycle type λ_x of an element $x \in S_\infty$ is well defined and it determines the conjugacy class of x in S_∞ completely. In other words, $y \in S_\infty$ is conjugate to x if and only if $\lambda_y = \lambda_x$. Let C_λ denote the corresponding conjugacy class in S_∞ , that is $C_\lambda = \{x \in S_\infty : \lambda_x = \lambda\}$. The intersection $C_\lambda \cap S_n$ is denoted by $C_\lambda(n)$ and we call it as the *n-part of C_λ* . By definition, $C_\lambda(n)$ is the set of all permutations x in S_n with $\lambda_{(x)} = \lambda(n)$, the *n-completion* of λ . Obviously, $C_\lambda(n)$ is non-empty if and only if $w(\lambda) \leq n$, and moreover, $C_\lambda(n)$ is a full conjugacy class in S_n . The proofs of these assertions are well known (see [FH]).

2.5 Indexing the B_n -double cosets

Each element x of S_{2n} has an associated undirected graph Γ_x and two permutations lie in the same B_n -double coset if and only if their associated graphs are isomorphic. Let us briefly explain this. Let $[2n]$ denote the set $\{1, 2, \dots, 2n\}$. Then the vertex set of the graph Γ_x of x is $V_x = \{v_i : i \in [2n]\}$, where $v_i := (i, x(i))$. The edge set E_x of the graph is a union of two disjoint sets, denoted respectively by R_x and B_x . The elements r_i , $i = 1, \dots, n$ of R_x are called the *straight edges*, and they are defined/denoted as in

$$R_x := \{r_i = (v_{2i-1} : v_{2i}) \mid i = 1, \dots, n\}.$$

The elements of B_x are defined by $b_i := (v_{x^{-1}(2i-1)} : v_{x^{-1}(2i)})$, $i = 1, \dots, n$, and they are called the *curved edges*.

Example 2.8. For $x = (1234)(5768)(9 \rightarrow 10)$ the associated graph Γ_x is depicted in Figure 2.8.

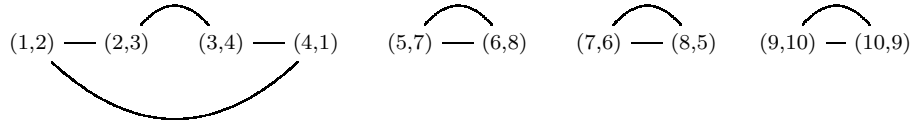


Figure 1: The graph of $x = (1234)(5768)(9 \rightarrow 10)$.

In general, each vertex on the graph Γ_x lies on exactly one straight and one curved edge, therefore, the connected components of Γ_x are of even size. If the lengths of the connected components of Γ_x are listed as $2s_1 \geq 2s_2 \geq \dots \geq 2s_k$, then $\mu_{(x)} := (s_1, \dots, s_k)$ is a partition of n . In Example 2.8, the partition is given by $\mu_{(x)} = (2, 1, 1, 1)$ and $l(\mu_{(x)}) = 4$. The partition $\mu_{(x)}$ is called the *coset type* of x , which is justified by the following well-known result:

Lemma 2.9 ([3]). Let $x, y \in S_{2n}$. Then

1. $\mu_{(x)} = \mu_{(y)}$ if and only if there is a graph isomorphism between Γ_x and Γ_y ;
2. $B_n x B_n = B_n y B_n$ if and only if there is a graph isomorphism between Γ_x and Γ_y .

As in the case of conjugacy classes, the partition $\mu_{(x)}$ is dependent on n . Along the similar lines, to characterize the B_∞ -double coset of an element $x \in S_\infty$, we have to “stabilize.” This is done by introducing the *stable coset type* μ_x of x : $\mu_x = \mu_{(x)} - (1^{l(\mu_{(x)})})$. The B_∞ -double cosets of two restricted permutations x, y in $S_{2\infty}$ are same if and only if $\mu_x = \mu_y$. The B_∞ -double coset of a stable partition is denoted by K_μ , and it consists of permutations $x \in S_\infty$ with $\mu_x = \mu$. The intersection $K_\mu \cap S_{2n}$ is denoted by $K_\mu(n)$ and it is called the n -part of K_μ . By definition, $K_\mu = \{x \in S_{2n} \mid \mu_{(x)} = \mu(n)\}$. Clearly, K_μ is a full B_n -double coset in S_{2n} . It is also clear that the $K_\mu(n)$ is non-empty if and only if $w(\mu) \leq n$. The proofs of these assertions are simple and recorded in [1].

The cardinality of a D -support stays constant on a B_∞ double coset. In fact, more is true:

Lemma 2.10. For any $x \in K_\mu$ the equality $|DS(x)| = 2|D(x)| = 2w(\mu)$ holds.

Proof. See [1]. □

Example 2.11. Let $\mu = (3, 2, 1)$. Then $w(\mu) = 9$ and thus $K_\mu \cap S_{2n}$ is non-empty if and only if $n \geq 9$. The permutations

$$x = (1357)(9 \rightarrow 11 \rightarrow 13)(15 \rightarrow 17)$$

and

$$y = (7 \rightarrow 9 \rightarrow 13)(5 \rightarrow 11 \rightarrow 12 \rightarrow 1 \rightarrow 3)(2 \rightarrow 14)(15 \rightarrow 17 \rightarrow 16)$$

are from K_μ , hence $B_n x B_n = B_n y B_n$ for all $n \geq 13$.

Next, we have a critical lemma about the cycles of a permutation $x \in S_{2n}$ and that of Γ_x .

Lemma 2.12. Let $x, x_1 \in S_{2n}$ be two permutations such that $x = c_1 x_1$, where c is a cycle satisfying

1. $S(c_1) \cap S(x_1) = \emptyset$;
2. $t(S(c_1)) \cap S(x) = \emptyset$.

Then $DS(c_1)$ is equal to the set of vertices of a connected component of Γ_x .

Proof. As $S(c_1) \cap S(x_1) = \emptyset$ it follows that $S(x) = S(c_1) \cup S(x_1)$ and hence $S(c_1) \cap t(S(c_1)) = \emptyset$. Therefore, no two elements of $S(c_1)$ are contained in the same couple. Let $c_1 = (i_1, \dots, i_k)$, hence $S(c_1) = \{i_1, \dots, i_k\}$. As $S(x) \cap t(S(c_1)) = \emptyset$, we have $x(t(i_j)) = c_1(t(i_j)) = t(i_j)$. So, there is an edge between $v_{i_{j-1}} = (i_{j-1}, x(i_{j-1})) = (i_{j-1}, i_j)$ and $(t(i_j), t(i_j)) = (t(i_j), x(t(i_j))) = v_{t(i_j)}$ as depicted in Figure 2.5.

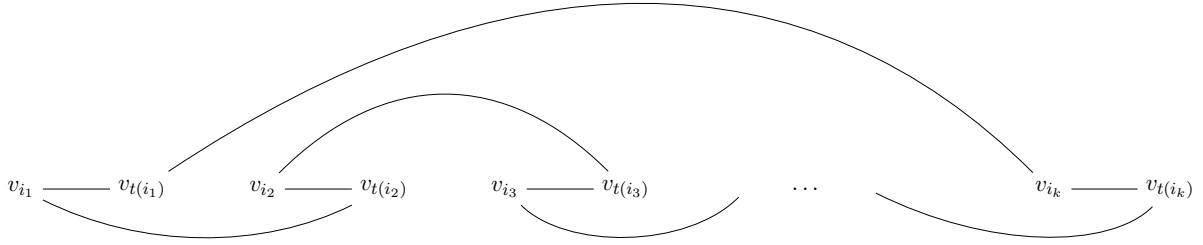


Figure 2: $v_{i_s} = (i_s, x(i_s))$ and $v_{t(i_s)} = (t(i_s), t(i_s))$.

□

Proposition 2.13. Let $\mu = (k_1, \dots, k_r)$ be a partition and $x \in K_\mu$ with $|S(x)| = w(\mu)$. Then $\lambda_x = \mu_x$.

Proof. Let $c_1 \dots c_s$ be the disjoint cycle decomposition of x . Here, we omit the cycles with one element. Set $l_1 = |S(c_1)| \geq \dots \geq l_s = |S(c_s)|$ so that $\lambda_x = (l_1 - 1, \dots, l_s - 1)$. Put $x_i = c_i^{-1} x$. We are first going to show that $x = c_i x_i$ satisfies the hypothesis of the Lemma 2.12. By definition, the cardinality of $D(x)$ is less than or equal to the number of couples D_i such that $D_i \cap S(x) \neq \emptyset$.

We know from Lemma 2.10 that $w(\mu) = |D(x)|$, hence $|S(x)| \geq |w(\mu)|$. As $|S(x)| = |w(\mu)|$, it follows that $S(x)$ contains elements from $|S(x)|$ many different couples, hence, the integers in $S(x)$ are not partners of each other. In other words, $t(S(c_i)) \cap S(x) = \emptyset$. Since $S(x) = S(c_i) \cup S(x_i)$, it follows in particular that $t(S(c_i)) \cap S(x_i) = \emptyset$.

Now, by applying Lemma 2.12, we see that the graph Γ_x has connected components C_1, \dots, C_k with the edge sets $DS(c_1), \dots, DS(c_r)$. The length of each C_i is given by $2|S(c_i)|$ for $i = 1, \dots, r$. The vertices on the rest of the graph are not contained in $DS(x)$, hence, the rest of Γ_x is a disjoint collection of cycles of length 2. Therefore, the stable coset type μ_x of x is $(l_1 - 1, \dots, l_s - 1)$, which is equal to the stable coset type λ_x . \square

Corollary 2.14. Let $x \in K_\mu$ be as in the hypothesis of Proposition 2.13. Then

$$S(x) \cap tS(x) = \emptyset.$$

Proof. See the second paragraph of the proof of Proposition 2.13. \square

Let K_μ^m denote the subset of K_μ consisting of permutations with support size m .

Corollary 2.15. For $m = w(\mu)$ the set K_μ^m is equal to full B_∞ -conjugacy class in $S_{2\infty}$.

Proof. Clearly, if $x \in K_\mu^m$, then any B_∞ -conjugate of x is contained in K_μ^m , also. We are going to show that there exists a single B_∞ -conjugacy class in K_μ^m . To this end, let x and y be two permutations from K_μ^m with disjoint cycle decompositions $x = x_1 \cdots x_r$ and $y = y_1 \cdots y_{r'}$. By our hypothesis and Proposition 2.13, we see that $r = r'$, and that $|S(x_i)| = |S(y_i)| = a_i$, $i = 1, \dots, r$. It follows that there exists a bijection $U : S(x) \rightarrow S(y)$ which restricts to bijections $S(x_i) \rightarrow S(y_i)$ for all $i = 1, \dots, r$, and hence, $U^{-1}yU = x$ holds. Indeed, we define U as follows. If $x = (i_1 \dots i_{r_1}) \cdots (i_{r_s} \dots i_{r_{s'}})$, and $y = (j_1 \dots j_{r_1}) \cdots (j_{r_s} \dots j_{r_{s'}})$ are the cycle decompositions of x and y , respectively, then U is defined by sending i_q to j_q . Now, we insist on the conditions: 1) U maps $t(i_q)$ to $t(j_q)$ (this makes sense by Corollary 2.14); 2) if i is not in $S(x, y) \cup tS(x, y)$, then $U(i) = i$. Clearly, $U \in B_\infty$, and the proof is complete. \square

Example 2.16. Let $\mu = (l_1 - 1, \dots, l_k - 1)$. Define $c_1 = (135 \dots 2l_1 - 1)$, $c_2 = (2l_1 + 1 \dots 2(l_1 + l_2) - 1)$, \dots , $c_k = (2(l_1 + \dots + l_{k-1} + 1) \dots 2(l_1 + \dots + l_k - 1))$. Then $c = c_1 \cdots c_k$ is in $K_\mu^{w(\mu)} \cap S_{2w(\mu)}$.

2.6 B_∞ -actions

In this subsection, following [1], we introduce two actions of $B_\infty \times B_\infty$ on $S_{2\infty} \times S_{2\infty}$ which turn out to be equivalent actions. Analogous actions are defined by Farahat and Higman in their classical paper [2] on the center of the symmetric group.

Let $(a, b) \in B_\infty \times B_\infty$ and $(x, y) \in S_{2\infty} \times S_{2\infty}$. Define

- The straightforward action: $(a, b) \cdot_s (x, y) = (axb^{-1}, ayb^{-1})$.
- The reverted action: $(a, b) \cdot_r (x, y) = (axb^{-1}, bya^{-1})$.

Clearly, the map $\phi : (S_{2\infty} \times S_{2\infty}, \cdot_s) \rightarrow (S_{2\infty} \times S_{2\infty}, \cdot_r)$ defined by

$$\phi((x, y)) = (x, y^{-1}) \tag{6}$$

is an equivariant bijection. In other words, $\phi(z \cdot_s (x, y)) = z \cdot_r \phi(x, y)$. If L is an orbit with respect to the straightforward action then $\varphi(L)$ is an orbit with respect to the reverted action. Thus, if O_s (respectively O_r) denotes the set of orbits of \cdot_s (respectively of \cdot_r) in $S_{2\infty}^2$, then ϕ induces a bijection from O_s to O_r . For an orbit (either straightforward, or reverted) L , the intersection $L \cap S_{2n}$ is denoted by $L(n)$.

Remark 2.17. Let L be an orbit of the reverted action. Then the integer $c_L := |S(xy)|$, called the *product-weight* of L , is independent of the element (x, y) chosen from L .

3 Gap

In this section we explain the gap in the proof of [Theorem 4.2 of [1]] in more detail. We start with paraphrasing its statement:

Theorem 3.1. Let μ, λ, ν be three partitions. Then there exists a polynomial $f_{\mu\lambda}^\nu(x) \in \mathbb{Q}[x]$ such that $b_{\mu\lambda}^\nu(n) = 2^{n-w(\nu)} f_{\mu\lambda}^\nu(n)$ for large enough $n \in \mathbb{Z}$.

Remark 3.2. In [1], the multiplicand $2^{n-w(\nu)} n!$ is missing, also.

Quoting from the proof of Theorem 4.2:

“Let λ, μ and ν be the stable coset types as given in the hypothesis. We already know that $b_{\lambda\mu}^\nu(n) = 0$ if $|\nu| > |\lambda| + |\mu|$. To prove the other statements, let \mathcal{A} denote the set of pairs $(x, y) \in S_\infty \times S_\infty$ satisfying $x \in K_\lambda, y \in K_\mu, xy \in K_\nu$. Then \mathcal{A} is stable under the reverted action of $B_\infty \times B_\infty$. Let $\mathcal{A}(n)$ denote the intersection $\mathcal{A} \cap (S_{2n} \times S_{2n})$. Hence, $b_{\lambda\mu}^\nu(n) = |\mathcal{A}(n)|/|K_\nu(n)|$.

Let $\{A_1, \dots, A_r\}$ denote the set of orbits of $B_\infty \times B_\infty$ in $\mathcal{A}(n)$. Then $b_{\lambda\mu}^\nu(n) = \frac{|\mathcal{A}(n)|}{|K_\nu(n)|} = \sum_{i=1}^r \frac{|A_i|}{|K_\nu(n)|}$.”

Then in [1], the proof is completed by using the polynomiality (in n) of the expressions $\frac{|A_i|}{|K_\nu(n)|}$ ($i = 1, \dots, r$), since the structure constant is equal to theirs sums. However, as n grows the number of orbits $\{A_1, \dots, A_r\}$ in $\mathcal{A}(n)$, hence the number of polynomial summands of the right hand side of $\frac{|\mathcal{A}(n)|}{|K_\nu(n)|} = \sum_{i=1}^r \frac{|A_i|}{|K_\nu(n)|}$, increases. Let $\{A_1, \dots, A_r\}$ be the set of all reverted orbits in $\mathcal{A}(n)$. Without loss of generality, we may assume that c_{A_1} is maximal. Let $(x, y) \in A_1$ and consider the element $(x_1, y_1) = ((2n+1)2n+2)x, y$, which is an element of $\mathcal{A}(n+1)$. However, since $|S(x_1 y_1)| = c_{L_1} + 2$, and c_{L_1} is maximal, (x_1, y_1) is not contained in any of the orbits A_1, \dots, A_r . This shows that the number r of orbits contained in $\mathcal{A}(n)$ gets bigger as n grows.

Remark 3.3. The set $\mathcal{A}(n)$ defined in quoted paragraph is nothing but $V(K_\mu(n) \times K_\lambda(n); K_\nu(n))$.

To fix the problem, we are going to replace the set $K_\nu(n)$ with K_ν^m , where $m = w(\nu)$.

4 Fix

Lemma 4.1. Let $L \in O_r$ be a reverted orbit, and let (x, y) be an arbitrary element from L . Define $m_L = m_L(x, y)$ (called the *magnitude* of the reverted orbit L), by the equation

$$2m_L = |S(xy)| + |t(S(xy))| + |DS(x)| + |DS(y)|. \quad (7)$$

Then $|CS(x', y')| \leq 2m_L$ for all $(x', y') \in L$, hence m_L is independent of the element (x, y) .

Proof. We start with a fixed element (x, y) from L . Then $|CS(x, y)| \leq 2m_L$. Let $(x', y') \in L$. Then $(axb^{-1}, bya^{-1}) = (x', y')$ for some $a, b \in B_\infty$. The equation $x'y' = axya^{-1}$ implies that $|S(xy)| = |S(x'y')|$. By Lemma 2.10, $|D(x)| = |D(x')|$ and $|D(y)| = |D(y')|$. Hence $|DS(x)| = |DS(x')|$, and $|DS(y)| = |DS(y')|$, proving that m_L is well defined. \square

Proposition 4.2. For any reverted orbit L , the intersection $L(m_L) := L \cap (S_{2m_L} \times S_{2m_L})$ is non-empty.

Our next observation is crucial for the proof of Proposition 4.2.

Lemma 4.3. Let $(x, y) \in S_{2n} \times S_{2n}$. Then there exists (x', y') in the reverted B_n -orbit of (x, y) such that $S(x', y') \subseteq CS(x', y') = CS(x, y)$. In particular $|S(x', y')| \leq |CS(x, y)| \leq 2m_L$.

Proof. Let $i \notin CS(x, y)$ and $y(i) = j \neq i$. We proceed by showing that there exists $(x_0, y_0) \in B_n(x, y)B_n$ such that

1. $x_0(i) = y_0(i) = i$;
2. $x_0 y_0 = xy$;
3. $CS(x_0, y_0) = CS(x, y)$.

Observe that once we prove the existence of such an element the result then follows by induction.

By Lemma 2.7, we have the following identities:

$$\begin{aligned} xy(i) &= i \\ x(j) &= i \\ y(t(i)) &= t(y(i)) = t(j) \\ x(t(j)) &= t(x(j)) \end{aligned}$$

On the other hand, there are two cases. Either $j = t(i)$, or not. For $j \neq t(i)$, we set $b = (ij)(t(i)t(j))$ and set (x_0, y_0) to be $(id, b) \cdot_r (x, y)$. Then (x_0, y_0) satisfies the properties 1.–3., listed above. Indeed, the first two properties are trivially satisfied. As $x_0 y_0 = xy$, it follows that $S(x_0 y_0) = S(xy)$. Hence, in order to show $CS(x, y) = CS(x_0, y_0)$, it suffices to show that $D(x, y) = D(x_0, y_0)$, or that $D(x) = D(x_0)$ and $D(y) = D(y_0)$. As x_0 and x are in the same B_∞ -double coset, it follows that $|D(x)| = |D(x_0)|$. So, it suffices to show that $D(x)$ is a subset of $D(x_0)$. To this end, let $\{r, t(r)\} \in D(x)$, hence $tx(r) \neq xt(r)$. Since $x_0(l) = x(l)$ for any $l \neq i, j, t(i), t(j)$, it follows that $x_0(r) = x(r) \neq x(t(r)) = x_0(t(r))$, hence $\{r, t(r)\} \in D(x_0)$. Finally, it is easy to check that the elements $\{i, t(i)\}$ and $\{j, t(j)\}$ are not contained in $D(x)$. Therefore, we see that $D(x) \subseteq D(x_0)$. Notice the same line of arguments apply to $D(y)$ and $D(y_0)$. In conclusion, when $j \neq t(i)$, we have $CS(x_0, y_0) = CS(x, y)$.

For the case $j = t(i)$ we use $b = (ij)$ to define $(x_0, y_0) := (id, b) \cdot_r (x, y)$. The rest of the argument is identical with that of the previous case, and therefore, the proof is complete. □

Now we are ready to prove Proposition 4.2.

Proof of Proposition 4.2. By using Lemma 4.3, we choose an element (x, y) from L such that $S(x, y) \subseteq CS(x, y)$. Recall that $m = |CS(x, y)| \leq m_L$. Listing the elements of $CS(x, y)$ in partners as follows $i_1, t(i_1), \dots, i_m, t(i_m)$ with $i_j < i_{j+1}$, we define an injection $u : CS(x, y) \rightarrow S_{2m_L}$ by sending i_j to $2j - 1$, and by sending $t(i_j)$ to $2j$ and keeping other integers stable. Obviously, u is an element of B_∞ and it satisfies $(u, u) \cdot_r (x, y) \in S_{2m_L}$, hence, the proof is complete. □

Corollary 4.4. Let μ, λ, ν be three partitions. Then $V = V(K_\mu \times K_\lambda; K_\nu^{w(\nu)})$ is a finite union of reverted B_∞ orbits.

Proof. Let $(a, b) \in B_\infty^2$ and $(x, y) \in V$. Clearly, $(a, b) \cdot_r (x, y) = (axb^{-1}, bya^{-1}) \in K_\mu \times K_\lambda$. As $xy \in K_\nu$, by Corollary 2.15, $axb^{-1} \cdot bya^{-1} = axya^{-1} \in K_\nu^{w(\nu)}$. Therefore, V is closed under the reverted action. Let L be the orbit containing $(x, y) \in V$. Then by using Lemma 2.10 and the fact that $xy \in K_\nu^{w(\nu)}$, we compute:

$$2m_L = |S(xy)| + |t(S(xy))| + |DS(x)| + |DS(y)| = 2(w(\nu) + w(\mu) + w(\lambda)).$$

Therefore, we conclude that the magnetite m_L does not depend on the orbit L , so we denote it by m_V . By Proposition 4.2, L contains an element from $S_{2m_V} \times S_{2m_V}$. By repeating this argument for each orbit L of V , we see that the orbits of V are parametrized by a subset of $S_{2m_V} \times S_{2m_V}$, hence there are only finitely many of them. \square

Let μ, λ , and ν be partitions. By definition, the integer $b_{\mu\lambda}^\nu(n)$ is defined as the coefficient of $\sum_{z \in K_\nu(n)} z$ in the product $\left(\sum_{x \in K_\mu(n)} x\right) \cdot \left(\sum_{y \in K_\lambda(n)} y\right)$. Equivalently, $b_{\mu\lambda}^\nu(n)$ is the number of couples $(x, y) \in K_\mu(n) \times K_\lambda(n)$ whose product xy lies in $K_\nu(n)$ divided by $|K_\nu(n)|$.

Lemma 4.5. Let μ, λ , and ν be partitions. Then

$$b_{\mu\lambda}^\nu(n) = \frac{|V(K_\mu(n) \times K_\lambda(n); K_\nu^{w(\nu)}(n))|}{|K_\nu^{w(\nu)}(n)|}. \quad (8)$$

Proof. Immediate from the fact that $K_\nu(n) = \bigcup_{k \geq 1} K_\nu^k(n)$ is a disjoint union. \square

Finally, we compute the size of the relevant orbit.

Lemma 4.6. Let L be a reverted orbit and ν be a partition. Then

1. There is a constant k_L such that $|L(n)| = \frac{(2^n n!)^2}{k(L)(2^{n-m_L}(n-m_L)!)}.$
2. There is a constant k_ν such that $|K_w(\mu)^\nu(n)| = \frac{2^n n!}{k_\nu p(2^{n-w(\nu)}(n-w(\nu)!))}.$

Proof. The proof of 1. follows closely the one that is presented in [1] with a minor modification. Let $L' = \phi^{-1}(L)$ be the straightforward orbit corresponding to L , where ϕ is as in (6). Then $|L(n)| = |L'(n)|$, therefore, it is enough to calculate the cardinality of the n -part of a straightforward orbit. We need to use the following result [Lemma 5.2, [1]] that $L'(n)$ is a straightforward $B_n \times B_n$ -orbit inside $S_{2n} \times S_{2n}$. Also, since by Proposition 4.2 $L(m_L) \subset S_{2n} \times S_{2n}$ is non-empty, there exists $(x', y') \in L'(n)$ such that x' and y' fixes integers i with $i > 2m_L$.

Observe that the stabilizer in $B_n \times B_n$ of such (x', y') splits:

$$\text{Stab}_{B_n \times B_n}((x', y')) = \text{Stab}_{B_{m_L} \times B_{m_L}}(z) \times \text{Stab}_{B_{n-m_L} \times B_{n-m_L}},$$

where B_{n-m_L} stands for the hyperoctahedral group on the set $[2n] \setminus [2m_L]$. It follows from definitions that $\text{Stab}_{B_{n-m_L} \times B_{n-m_L}} \cong B_{n-m_L}$. Therefore, if $k(L)$ denotes the number of elements of the first factor, then the number of elements of the orbit $L(n)$ is $(2^n n!)^2 / k(L) 2^{n-m_L} (n-m_L)!$.

For 2., we use the idea that is used in [2]. Let $x \in K^{w(\nu)}(n)$ so that $x \in S_{2w(\nu)}$ (see Example 2.16). Then by Corollary 2.15 $K_\mu^{w(\nu)}(n)$ is equal to the B_n conjugacy class of x in S_{2n} . So we need to calculate the B_n conjugacy class of x . It is equal to $|B_n|/|C_{B_n}(x)|$, where $C_{B_n}(x)$ is the centralizer of x in B_n which is the intersection of B_n with $C_{S_{2n}}(x)$. On the other hand, $C_{S_{2n}}(x)$ is equal to the direct product of the centralizer of x in $S_{2w(\nu)}$ and the symmetric group complement to $S_{2w(\nu)}$ in S_{2n} . By the same reasoning, the centralizer of x in B_n is the direct product of the centralizer of x in $B_{w(\nu)}$ and the hyperoctahedral subgroup in the symmetric group that is complement to $S_{2w(\nu)}$ in S_{2n} , which is isomorphic to $B_{n-w(\nu)}$. Then, if k_ν denotes the number of elements in the centralizer of x in $B_{w(\nu)}$ the number of elements in the centralizer of x in B_n is equal to $k_\nu |B_{n-w(\nu)}|$. The result now follows. \square

We are ready to fill the gap.

Proof of Theorem 3.1. Let L_1, \dots, L_r be the list of all reverted orbits contained in $V = V(K_\mu \times K_\lambda; K_\nu^{w(\nu)})$. By Corollary 4.4, we know that

$$V(n) := V(K_\mu(n) \times K_\lambda(n); K_\nu^{w(\nu)}(n)) = L_1(n) \cup \dots \cup L_r(n).$$

Thus, by Lemma 4.5, it is enough to compute $b_{\mu\lambda}^\nu(n) = \sum_{i=1}^r \frac{|L_i(n)|}{|K_\nu^{w(\nu)}(n)|}$. For $i = 1, \dots, r$

$$\begin{aligned} \frac{|L_i|}{|K_\nu^{w(\nu)}(n)|} &= \frac{(2^n n!)^2}{k(L_i)(2^{n-m_L}(n-m_L)!)} \cdot \frac{k_\nu(2^{n-w(\nu)}(n-w(\nu))!)}{2^n n!} \\ &= 2^{n-w(\nu)} n! \frac{2^{m_{L_i}} \left(n(n-1) \cdots (n-m_{L_i}+1) \right) \left(n(n-1) \cdots (n-w(\nu)+1) \right)}{k(L_i)}, \end{aligned}$$

where k_ν and $k(L_i)$ are as in Lemma 4.6. It follows that, for $i = 1, \dots, r$, the expressions $f_i(n) := \frac{|L_i(n)|}{|K_\nu^{w(\nu)}(n)| 2^{n-w(\nu)} n!}$ are polynomials in n , hence, so is their sum. Since $b_{\mu\lambda}^\nu(n) = 2^{n-w(\nu)} n! \sum_{i=1}^r f_i(n)$, the proof is complete. \square

Remark 4.7. Observe that if we normalize the characteristic function $\chi_i(n)$ of the double coset \bar{x}_i (using the notation of Introduction) by $\chi'_i(n) = \frac{1}{2^{n-1}n!} \chi_i(n)$, then the corresponding structure constants $b_{ij}^k(n)$'s become polynomials in n .

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